

THE GEOMETRY OF BORDER BASES

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ABSTRACT. In this paper we continue the study of the border basis scheme we started in [16]. The main topic is the construction of various explicit flat families of border bases. To begin with, we cover the punctual Hilbert scheme $\text{Hilb}^\mu(\mathbb{A}^n)$ by border basis schemes and work out the base changes. This enables us to control flat families obtained by linear changes of coordinates. Next we provide an explicit construction of the principal component of the border basis scheme, and we use it to find flat families of maximal dimension at each radical point. Finally, we connect radical points to each other and to the monomial point via explicit flat families on the principal component.

1. INTRODUCTION

*Nobody goes there anymore
because it's too crowded.*
(Yogi Berra)

Border bases schemes have recently become an active area of research, as evidenced by a list of references which is getting quite crowded, e.g. [1], [5], [9], [11], [13], [16], [17], [18], and [20], to mention just a few contributions of the last years. What is the reason for this spurt of activity?

In our opinion there are several reasons. One of them is that border bases enjoy a degree of numerical stability which, in contrast, Gröbner bases don't. This has proven useful for dealing with empirical polynomials constructed from measured data (see for instance [13] and [21]) and has even led to actual industrial applications. But the most relevant aspect for our topic is that border basis schemes provide a very concrete and easily accessible way to parametrize 0-dimensional polynomial ideals. They can be viewed as open affine subschemes of the corresponding Hilbert schemes which can be described by simple, explicit polynomial equations (see for instance [16] and [19]).

This brings us to our first contribution: by constructing explicit matrices describing the change of basis between one border basis scheme and another, we obtain a direct construction of the punctual Hilbert scheme $\text{Hilb}^\mu(\mathbb{A}^n)$ (see [8] and [10]) which uses neither A. Grothendieck's Grassmannian variety technique nor any arguments involving representation of functors.

In the paper [20] it was shown that in some cases border bases schemes can be described via suitable Gröbner basis schemes. This is an important fact, since the description of Gröbner basis schemes requires fewer indeterminates and fewer equations, and motivates the strategy used in Section 1 to treat the cases of border basis and Gröbner basis schemes simultaneously.

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Another *driving force* for writing this paper was the search for suitable flat deformations of border bases. We have seen in [16] that flat deformations of border bases are the same as rational curves on the border basis scheme $\mathbb{B}_{\mathcal{O}}$. Therefore we want to construct explicit rational curves on $\mathbb{B}_{\mathcal{O}}$. However, in contrast to the Hilbert scheme, we have the problem that a flat family whose general fibre is an ideal corresponding to a point in $\mathbb{B}_{\mathcal{O}}$, i.e. an ideal having an \mathcal{O} -border basis, can have a special fibre which doesn't (see for instance [16], Example 3.9). Constructing many flat families of border bases could be one way to attack the hitherto unsolved problem of the connectedness of the border basis scheme (see [20], Question 2). In various parts of this paper we construct a number of such flat families, in addition to the ones we found by homogenization in [16]: families obtained from linear changes of coordinates, from local parametrizations of the principal component, and from distractions.

Let us describe the contents of the paper in more detail. In Section 2 we introduce pseudo order ideals, pseudo borders and pseudo border bases (see Definition 2.1). Using them, we are not only able to treat the cases of order ideals with their borders and of σ -cornercuts with their corners simultaneously, but also their isomorphic images under a linear change of coordinates. After constructing a moduli space for pseudo border bases (cf. Prop. 2.2) and showing that it comes equipped with a universal flat family and has the expected properties (cf. Prop. 2.5), we arrive at our first main result. In Theorem 2.7 we construct an explicit isomorphism between the open subsets of two pseudo border basis schemes corresponding to the ideals which have pseudo border bases with respect to both pseudo order ideals.

As mentioned above, this yields an explicit description of the glueing of border basis schemes necessary to build the corresponding punctual Hilbert scheme (see Remark 2.8). Another application of the theorem is the possibility to characterize when a linear change of coordinates produces an ideal which has again a pseudo border basis with respect to the same pseudo order ideal (see Prop. 2.11). Thus we can use linear changes of coordinates to construct explicit flat families of border bases (see Proposition 2.12). Moreover, we show that, in the \mathcal{O} -border basis setting, generic linear changes of coordinates lead to ideals which have \mathcal{O} -border bases again (see Corollary 2.14) and conclude quippingly that in border basis theory there is *no gin*.

In Section 3 we start our exploration of the principal component of the border basis scheme (see Definition 3.1). The first step is Theorem 3.6 where we provide explicit equations defining this scheme. Next we show in Proposition 3.8 that our construction yields the same result as the one given in [6] and [7], but uses a much smaller number of algebra generators. This has the obvious advantage that one can turn our description into an algorithm for computing the vanishing ideal of the principal component (see Prop. 3.9), and that one can use this algorithm to check whether a given border basis scheme is irreducible.

In Section 4 we use the principal component $\mathbb{C}_{\mathcal{O}}$ of $\mathbb{B}_{\mathcal{O}}$ to construct more explicit flat families of border bases. More precisely, we construct explicit local parameters at a radical point of $\mathbb{C}_{\mathcal{O}}$. The idea of this construction is to use the complete intersection representation of a radical ideal provided by the Shape Lemma (c.f. [15], Theorem 3.7.25) and to apply the techniques of Section 1 to it. The resulting Theorem 4.2 not only recovers well-known facts, e.g. that $\mathbb{C}_{\mathcal{O}}$ is a rational variety and non-singular at its radical points, but it gives us an explicit parametrization of

an open neighborhood of every radical point. We use this theorem in several ways: to connect two radical points on $\mathbb{C}_{\mathcal{O}}$ via a sequence of two explicit flat families (cf. Remark 4.4), and to combine these families with a distraction to connect every radical point of $\mathbb{C}_{\mathcal{O}}$ to the monomial point (cf. Remark 4.7).

Unless explicitly stated otherwise, we use the notation and the definitions introduced in [14] and [15].

2. CHANGE OF BASIS

Let K be a field, let $P = K[x_1, \dots, x_n]$, let $I \subset P$ be a 0-dimensional ideal, and let $\mu = \dim_K(P/I)$. As mentioned in the introduction, it is our goal to construct explicit flat families of ideals having I as one of their fibers. A natural approach is to perform linear changes of coordinates, i.e. K -algebra isomorphisms $\varphi : P \rightarrow P$ mapping the indeterminates to (not necessarily homogeneous) linear polynomials. If the ideal I has a border basis with respect to some order ideal, the same is not always true for the ideal $\varphi(I)$. Therefore one of the ideas of the following construction is to keep track when a linear change of coordinates preserves the property that I has a border basis with respect to a given order ideal.

Recall that a finite set of terms \mathcal{O} in \mathbb{T}^n is called an **order ideal** if it is closed under forming divisors, i.e. if $t \in \mathcal{O}$ and $t' \mid t$ imply $t' \in \mathcal{O}$. The set of terms $\partial\mathcal{O} = (x_1\mathcal{O} \cup \dots \cup x_n\mathcal{O}) \setminus \mathcal{O}$ is called the **border** of \mathcal{O} . The definition of an order ideal implies that the set $\mathbb{T}^n \setminus \mathcal{O}$ is a monomial ideal. We denote the set of monomial generators of this monomial ideal by $c\mathcal{O}$ and call it the **corner set** of \mathcal{O} . Let σ be a term ordering. The order ideal \mathcal{O} is called a **σ -cornercut** if $b >_{\sigma} t$ for all $b \in c\mathcal{O}$ and all $t \in \mathcal{O}$. Notice that this implies $b >_{\sigma} t$ for all $b \in \partial\mathcal{O}$ and all $t \in \mathcal{O}$.

The following definition is manufactured in such a way that we can treat the cases of the border basis scheme and the Gröbner basis scheme simultaneously.

Definition 2.1. Let $\varphi : P \rightarrow P$ be a linear change of coordinates, and let σ be a term ordering.

- (a) Let \mathcal{P} and $b\mathcal{P}$ be sets of polynomials in P . Then \mathcal{P} is called a **pseudo order ideal** and $b\mathcal{P}$ is called the **pseudo border** of \mathcal{P} if one of the following two cases occurs:
 - (i) \mathcal{P} is the image of an order ideal \mathcal{O} under φ , and $b\mathcal{P}$ is the corresponding image of the border of \mathcal{O} ;
 - (ii) \mathcal{P} is the image of a σ -cornercut \mathcal{O} under φ , and $b\mathcal{P}$ is the corresponding image of the corner set $c\mathcal{O}$.
- (b) Let \mathcal{P} be a pseudo order ideal, and let I be an ideal in P such that the residue classes of the elements of \mathcal{P} form a K -vector space basis of P/I . In this case we simply say that \mathcal{P} is a **basis modulo I** .
- (c) Let $\mathcal{P} = \{t_1, \dots, t_{\mu}\}$ be a pseudo order ideal in P , let $b\mathcal{P} = \{b_1, \dots, b_{\nu}\}$ be its pseudo border, and for $j = 1, \dots, \nu$ let $g_j = b_j - \sum_{i=1}^{\mu} \gamma_{ij} t_i$ with $\gamma_{ij} \in K$. Then the set $G = \{g_1, \dots, g_{\nu}\}$ is called a **pseudo \mathcal{P} -border prebasis**.
- (d) A pseudo \mathcal{P} -border prebasis $G = \{g_1, \dots, g_{\nu}\}$ is called a **pseudo \mathcal{P} -border basis** if \mathcal{P} is a basis modulo the ideal (g_1, \dots, g_{ν}) .

- (e) Let $\alpha = \#(b\mathcal{P})$, and let $C = (c_{ij})$ be a matrix of indeterminates of size $\mu \times \alpha$. For $j = 1, \dots, \alpha$, we form the polynomials $g_j = b_j - \sum_{i=1}^{\mu} c_{ij}t_i$. Then $G = \{g_1, \dots, g_{\alpha}\}$ is called the **generic pseudo \mathcal{P} -border prebasis**.

In the following we shall assume the setting and notation of this definition. We start our investigation with the following fact.

Proposition 2.2. *Let $\mathcal{P} = \{t_1, \dots, t_{\mu}\}$ be a pseudo order ideal in P , let $b\mathcal{P} = \{b_1, \dots, b_{\alpha}\}$ be its pseudo border, and let $C = (c_{ij})$ be a matrix of indeterminates of size $\mu \times \alpha$. There exists an ideal $I(\mathbb{B}_{\mathcal{P}})$ in $K[c_{11}, \dots, c_{\mu\alpha}]$ such that the ring $B_{\mathcal{P}} = K[c_{11}, \dots, c_{\mu\alpha}]/I(\mathbb{B}_{\mathcal{P}})$ is the coordinate ring of an affine scheme $\mathbb{B}_{\mathcal{P}}$ whose K -rational points are in one-to-one correspondence with the ideals I in P for which \mathcal{P} is a basis modulo I .*

Proof. If $\mathcal{P} = \mathcal{O}$ is an order ideal of terms and $b\mathcal{P} = \partial\mathcal{O}$ its border, the claim follows from [15], Theorem 6.4.30. If $\mathcal{P} = \mathcal{O}$ is a σ -cornercut for some term ordering σ and $b\mathcal{P} = c\mathcal{O}$, the claim follows from [20], Proposition 3.11. Given an ideal $I \subset P$ such that \mathcal{O} is a basis modulo I , let C_I be the matrix obtained by substituting the entries c_{ij} of C with the coordinates of the point in the scheme $\mathbb{B}_{\mathcal{O}}$ corresponding to I . We observe that, in both cases, we have

$$b\mathcal{O} = \mathcal{O} \cdot C_I \pmod{I} \quad (1)$$

Next, let \mathcal{O} be an order ideal satisfying one of the preceding two conditions, let $\varphi : P \rightarrow P$ be a linear change of coordinates, and let $\mathcal{P} = \varphi(\mathcal{O})$. By definition, we have $b\mathcal{P} = \varphi(b\mathcal{O})$. We apply (1) to \mathcal{O} and $b\mathcal{O}$, and we get

$$\varphi(b\mathcal{O}) = \varphi(\mathcal{O}) \cdot C_I \pmod{\varphi(I)} \quad (2)$$

for all ideals I in P such that \mathcal{O} is a basis modulo I , and therefore

$$b\mathcal{P} = \mathcal{P} \cdot C_I \pmod{\varphi(I)} \quad (3)$$

Now let J be an ideal in P such that \mathcal{P} is a basis modulo J . Then $\varphi^{-1}(J)$ is an ideal such that \mathcal{O} is a basis modulo $\varphi^{-1}(J)$, and we can use (2) and (3). The fact that $J = \varphi(\varphi^{-1}(J))$ implies $b\mathcal{P} = \mathcal{P} \cdot C_{\varphi^{-1}(J)}$ modulo J . Hence, if we define $\mathbb{B}_{\mathcal{P}} = \mathbb{B}_{\mathcal{O}}$ and $I(\mathbb{B}_{\mathcal{P}}) = I(\mathbb{B}_{\mathcal{O}})$, the ideals J in P such that \mathcal{P} is a basis modulo J correspond one-to-one to the ideals $I = \varphi^{-1}(J)$ in P such that \mathcal{O} is a basis modulo I . \square

In the setting of this proposition, we introduce further terminology.

Definition 2.3. As above, let \mathcal{P} be a pseudo order ideal, $b\mathcal{P}$ its pseudo border, and $\alpha = \#(b\mathcal{P})$.

- (a) The scheme $\mathbb{B}_{\mathcal{P}}$ is called the **\mathcal{P} -basis scheme**.
- (b) Given an ideal I in the polynomial ring P such that \mathcal{P} is a basis modulo I , we write $b\mathcal{P} = \mathcal{P} \cdot C_I$ modulo I . Then the matrix $C_I \in \text{Mat}_{\alpha\mu}(K)$ and the point c_I whose coordinates are the entries of C_I are said to **represent I** in $\mathbb{B}_{\mathcal{P}}$.
- (c) Let $G = \{g_1, \dots, g_{\alpha}\}$ be the generic pseudo \mathcal{P} -border prebasis, and let

$$U_{\mathcal{P}} = K[x_1, \dots, x_n, c_{11}, \dots, c_{\mu\alpha}]/(I(\mathbb{B}_{\mathcal{P}}) + (g_1, \dots, g_{\alpha}))$$

Then the natural homomorphism of K -algebras $\Phi : B_{\mathcal{P}} \rightarrow U_{\mathcal{P}}$ is called the **universal \mathcal{P} -basis family**.

Remark 2.4. Let us point out one fact that follows from the proof of the preceding proposition: given a linear change of coordinates $\varphi : P \rightarrow P$, the matrix C_I which represents an ideal I in $\mathbb{B}_{\mathcal{O}}$ also represents $\varphi(I)$ in $\mathbb{B}_{\varphi(\mathcal{O})}$.

As in the usual border basis theory, a central property of the universal family is that it is free with basis \mathcal{P} . This is the main part of the following proposition.

Proposition 2.5. *As above, let \mathcal{P} be a pseudo order ideal, let $b\mathcal{P}$ be its pseudo border, let $G = \{g_1, \dots, g_\alpha\}$ be the generic pseudo \mathcal{P} -border prebasis, and let $\Phi : B_{\mathcal{P}} \rightarrow U_{\mathcal{P}}$ be the universal \mathcal{P} -basis family.*

- (a) *The residue classes of the elements of \mathcal{P} are a $B_{\mathcal{P}}$ -module basis of $U_{\mathcal{P}}$.*
- (b) *Via Φ , the ring $B_{\mathcal{P}}$ is a free summand of $U_{\mathcal{P}}$. In particular, the map Φ is injective and $B_{\mathcal{P}}$ can be seen as a subring of $U_{\mathcal{P}}$.*
- (c) *The rewriting rules given by the tuple (g_1, \dots, g_α) yield an explicit division algorithm with the following properties: for every polynomial f in $K[x_1, \dots, x_n, c_{11}, \dots, c_{\mu\alpha}]$, it produces a polynomial f' which is a linear combination of the elements in \mathcal{P} with coefficients in $K[c_{11}, \dots, c_{\mu\alpha}]$. The residue classes of these coefficients in $B_{\mathcal{P}}$ are uniquely determined and do not depend on the ordering of (g_1, \dots, g_α) .*

Proof. First we show (a). Let \mathcal{O} be an order ideal in \mathbb{T}^n and $\varphi : P \rightarrow P$ a linear change of coordinates such that $\mathcal{P} = \varphi(\mathcal{O})$. By [16], Theorem 3.4 and [20], Theorem 2.9, respectively, the set \mathcal{O} is a $B_{\mathcal{O}}$ -module basis of $U_{\mathcal{O}}$. Now the claim follows from the fact that we used $I(\mathbb{B}_{\mathcal{P}}) = I(\mathbb{B}_{\mathcal{O}})$ in Proposition 2.2.

Since (b) follows immediately from (a), it remains to prove (c). We denote the extension of φ to $P[c_{ij}]$ by $\tilde{\varphi}$, and again we write $\mathcal{P} = \varphi(\mathcal{O})$ with an order ideal \mathcal{O} . To define the algorithm for dividing f by (g_1, \dots, g_α) , we use the usual border division algorithm (cf. [15], Proposition 6.4.11) to divide $\tilde{\varphi}^{-1}(f)$ by $(\tilde{\varphi}^{-1}(g_1), \dots, \tilde{\varphi}^{-1}(g_\alpha))$. At the end we apply $\tilde{\varphi}$ to the resulting representation of $\tilde{\varphi}^{-1}(f)$ as a $K[c_{ij}]$ -linear combination of the elements of \mathcal{O} and obtain the desired $K[c_{ij}]$ -linear combination of elements of \mathcal{P} . The uniqueness of this representation is a consequence of (a). \square

Next we let I be an ideal in P such that two pseudo order ideals \mathcal{P} and \mathcal{P}' are bases modulo I . Suppose that I is represented by a matrix C_I in $\mathbb{B}_{\mathcal{P}}$ and a matrix D_I in $\mathbb{B}_{\mathcal{P}'}$. What is the relation between C_I and D_I ? In the following we examine this question.

Remark 2.6. Let \mathcal{P} and \mathcal{P}' be two pseudo order ideals for which $\mu = \#\mathcal{P} = \#\mathcal{P}'$. The scheme $\mathbb{B}_{\mathcal{P}}$ parametrizes ideals I in P such that \mathcal{P} is a basis modulo I . It contains a Zariski-open subset $\mathbb{B}_{\mathcal{P}, \mathcal{P}'}$ which parametrizes the ideals I such that also \mathcal{P}' is a basis modulo I . Similarly, there is an open subset $\mathbb{B}_{\mathcal{P}', \mathcal{P}}$ of $\mathbb{B}_{\mathcal{P}'}$ which parametrizes the ideals I for which both \mathcal{P} and \mathcal{P}' are bases modulo I .

It is known that $\mathbb{B}_{\mathcal{P}}$ and $\mathbb{B}_{\mathcal{P}'}$ are open subschemes of the same punctual Hilbert scheme $\text{Hilb}^\mu(\mathbb{A}^n)$ (see for instance [19], Chapter 18). Their intersection includes a non-empty open subset of the principal component (i.e. the component corresponding to radical ideals, cf. Section 3). Consequently, the open subsets $\mathbb{B}_{\mathcal{P}, \mathcal{P}'}$ of $\text{Hilb}^\mu(\mathbb{A}^n)$, which are equal by definition, are not empty.

In the following, we let \mathcal{P} and \mathcal{P}' be two pseudo order ideals such that $\mu = \#\mathcal{P} = \#\mathcal{P}'$. Let $\alpha = \#(b\mathcal{P})$ and $\alpha' = \#(b\mathcal{P}')$ be the cardinalities of their pseudo borders, let $C = (c_{ij})$ be a matrix of indeterminates of size $\mu \times \alpha$, and let $D = (d_{ij})$

be a matrix of indeterminates of size $\mu \times \alpha'$. According to Proposition 2.5.c, there exist matrices M_C and N_C over $K[c_{ij}]$ such that

$$\mathcal{P}' = \mathcal{P} \cdot M_C \quad b\mathcal{P}' = \mathcal{P} \cdot N_C \quad (4)$$

are matrix equalities which hold over $U_{\mathcal{P}}$. Similarly, there exist matrices M_D and N_D over $K[d_{ij}]$ such that

$$\mathcal{P} = \mathcal{P}' \cdot M_D \quad b\mathcal{P} = \mathcal{P}' \cdot N_D \quad (5)$$

are matrix equalities which hold over $U_{\mathcal{P}'}$.

Theorem 2.7. (Base Change for Pseudo Border Basis Schemes)

Assume that we are in the setting described above.

- (a) *The set $\mathbb{B}_{\mathcal{P}, \mathcal{P}'}$ is the open subset of $\mathbb{B}_{\mathcal{P}}$ defined by $\det(M_C) \neq 0$, and the set $\mathbb{B}_{\mathcal{P}', \mathcal{P}}$ is the open subset of $\mathbb{B}_{\mathcal{P}'}$ defined by $\det(M_D) \neq 0$.*
- (b) *The natural maps defining the identity map between $\mathbb{B}_{\mathcal{P}, \mathcal{P}'}$ and $\mathbb{B}_{\mathcal{P}', \mathcal{P}}$ in terms of their respective systems of coordinates are given parametrically by*

$$D = M_C^{-1} \cdot N_C \quad \text{and} \quad C = M_D^{-1} \cdot N_D$$

Proof. Claim (a) follows immediately from (4) and (5). Now we prove claim (b). By the definition of the generic pseudo border prebases, we have the equalities

$$b\mathcal{P} = \mathcal{P} \cdot C \quad \text{and} \quad b\mathcal{P}' = \mathcal{P}' \cdot D \quad (6)$$

In the following we work over the open set where both M_C and M_D are invertible, i.e. where both systems of coordinates (c_{ij}) and (d_{ij}) apply. Combining the second equality in (6) with the first in (4), we get

$$b\mathcal{P}' = \mathcal{P} \cdot M_C \cdot D$$

Now we compare this to the second equality in (4) and use the uniqueness implied by the fact that \mathcal{P} is a $B_{\mathcal{P}}$ -basis of $U_{\mathcal{P}}$ to get

$$M_C \cdot D = N_C$$

This implies the first claim in (b). The second claim follows by interchanging the roles of \mathcal{P} and \mathcal{P}' . \square

The formulas given in this theorem can be used as follows to give an explicit construction for the punctual Hilbert scheme.

Remark 2.8. (Glueing Border Basis Schemes)

Given an integer $\mu > 0$, it is well-known that there exists a scheme, called the *punctual Hilbert scheme* and denoted by $\text{Hilb}^{\mu}(\mathbb{A}^n)$ which parametrizes all 0-dimensional ideals I in P such that $\dim_K(P/I) = \mu$. For an introduction to punctual Hilbert schemes, see for instance [19] and its bibliography. Here we just want to point out that Theorem 2.7 allows to construct $\text{Hilb}^{\mu}(\mathbb{A}^n)$ very explicitly.

To explain the method, we use the following example. Let $n = 2$ and $\mu = 4$, i.e. we want to parametrize ideals which correspond to 0-dimensional subschemes of \mathbb{A}^2 of length four. There exist exactly five order ideals which can serve as a K -basis of P/I , namely

$$\begin{aligned} \mathcal{O}_1 &= \{1, x, x^2, x^3\}, & \mathcal{O}_2 &= \{1, y, y^2, y^3\}, & \mathcal{O}_3 &= \{1, x, y, x^2\}, \\ \mathcal{O}_4 &= \{1, x, y, y^2\}, & \text{and} & & \mathcal{O}_5 &= \{1, x, y, xy\}. \end{aligned}$$

Let $I_1 = (x^4, y)$, $I_2 = (x, y^4)$, $I_3 = (xy, y^2, x^3)$, $I_4 = (x^2, xy, y^3)$, and $I_5 = (x^2, y^2)$. It is clear that, for $i = 1, \dots, 5$, among the five order ideals the only basis

modulo I_i is \mathcal{O}_i . Therefore all the border basis schemes of all five order ideals are needed to cover $\text{Hilb}^4(\mathbb{A}^2)$ with affine open sets. The scheme $\text{Hilb}^4(\mathbb{A}^2)$ can be constructed explicitly by glueing the schemes $\mathbb{B}_{\mathcal{O}_1}, \dots, \mathbb{B}_{\mathcal{O}_5}$ via the isomorphisms given in Theorem 2.7.

Furthermore, we note that, for $i = 1, \dots, 4$, the set \mathcal{O}_i is a σ_i -cornercut for a suitable term ordering σ_i . However, this is not the case for \mathcal{O}_5 . Consequently, we have $\mathbb{B}_{\mathcal{O}_i} = \mathbb{G}_{\sigma_i, \mathcal{O}_i}$ for $i = 1, \dots, 4$ (see [20], Proposition 3.11). On the other hand, a direct calculation yields $\dim(\mathbb{B}_{\mathcal{O}_5}) = 8$, as expected, but $\dim(\mathbb{G}_{\sigma, \mathcal{O}_5}) \leq 7$ for every term ordering σ . This implies that we cannot cover $\text{Hilb}^4(\mathbb{A}^2)$ with open sets associated to Gröbner basis schemes. Using Gröbner basis schemes, we merely get a stratification of the Hilbert scheme.

The following example illustrates the theorem.

Example 2.9. In the ring $P = \mathbb{Q}[x, y]$ we consider the two order ideals $\mathcal{O} = \{1, y, x, xy\}$ and $\mathcal{O}' = \{1, x, x^2, x^3\}$. Our goal is to find the transformation matrices mentioned in the preceding theorem and to verify the equations given in its proof. (We have ordered all sets and tuples of terms according to **DegRevLex**.)

First of all, we represent \mathcal{O}' in terms of \mathcal{O} , modulo the generic \mathcal{O} -border basis $G = \{g_1, g_2, g_3, g_4\}$ where

$$\begin{aligned} g_1 &= y^2 - c_{11} - c_{21}y - c_{31}x - c_{41}xy \\ g_2 &= x^2 - c_{12} - c_{22}y - c_{32}x - c_{42}xy \\ g_3 &= xy^2 - c_{13} - c_{23}y - c_{33}x - c_{43}xy \\ g_4 &= x^2y - c_{14} - c_{24}y - c_{34}x - c_{44}xy \end{aligned}$$

The result is

$$\mathcal{O}' = \mathcal{O} \cdot M_C = \mathcal{O} \cdot \begin{pmatrix} 1 & 0 & c_{12} & c_{12}c_{32} + c_{14}c_{42} \\ 0 & 0 & c_{22} & c_{22}c_{32} + c_{24}c_{42} \\ 0 & 1 & c_{32} & c_{12} + c_{32}^2 + c_{34}c_{42} \\ 0 & 0 & c_{42} & c_{22} + c_{32}c_{42} + c_{42}c_{44} \end{pmatrix}$$

Similarly, we represent $\partial\mathcal{O}'$ in terms of \mathcal{O} and find

$$\partial\mathcal{O}' = \mathcal{O} \cdot N_C = \mathcal{O} \cdot \begin{pmatrix} 0 & 0 & c_{14} & c_{12}c_{34} + c_{14}c_{44} & h_1 \\ 1 & 0 & c_{24} & c_{22}c_{34} + c_{24}c_{44} & h_2 \\ 0 & 0 & c_{34} & c_{32}c_{34} + c_{34}c_{44} + c_{14} & h_3 \\ 0 & 1 & c_{44} & c_{34}c_{42} + c_{44}^2 + c_{24} & h_4 \end{pmatrix}$$

where

$$\begin{aligned} h_1 &= c_{12}c_{32}^2 + c_{14}c_{32}c_{42} + c_{12}c_{34}c_{42} + c_{14}c_{42}c_{44} + c_{12}^2 + c_{14}c_{22} \\ h_2 &= c_{22}c_{32}^2 + c_{24}c_{32}c_{42} + c_{22}c_{34}c_{42} + c_{24}c_{42}c_{44} + c_{12}c_{22} + c_{22}c_{24} \\ h_3 &= c_{32}^3 + 2c_{32}c_{34}c_{42} + c_{34}c_{42}c_{44} + 2c_{12}c_{32} + c_{22}c_{34} + c_{14}c_{42} \\ h_4 &= c_{32}^2c_{42} + c_{34}c_{42}^2 + c_{32}c_{42}c_{44} + c_{42}c_{44}^2 + c_{22}c_{32} + c_{12}c_{42} + c_{24}c_{42} + c_{22}c_{44} \end{aligned}$$

On the other hand, in terms of the coordinates d_{ij} we have

$$M_D = \begin{pmatrix} 1 & d_{11} & 0 & d_{12} \\ 0 & d_{21} & 1 & d_{22} \\ 0 & d_{31} & 0 & d_{32} \\ 0 & d_{41} & 0 & d_{42} \end{pmatrix} \quad \text{and} \quad N_D = \begin{pmatrix} k_1 & 0 & \ell_1 & d_{13} \\ k_2 & 0 & \ell_2 & d_{23} \\ k_3 & 1 & \ell_3 & d_{33} \\ k_4 & 0 & \ell_4 & d_{43} \end{pmatrix}$$

where

$$\begin{aligned}
k_1 &= d_{11}^2 + d_{12}d_{21} + d_{13}d_{31} + d_{14}d_{41} \\
k_2 &= d_{11}d_{21} + d_{21}d_{22} + d_{23}d_{31} + d_{24}d_{41} \\
k_3 &= d_{11}d_{31} + d_{21}d_{32} + d_{31}d_{33} + d_{34}d_{41} \\
k_4 &= d_{11}d_{41} + d_{21}d_{42} + d_{31}d_{42} + d_{41}d_{44} \\
\ell_1 &= d_{11}d_{12} + d_{12}d_{22} + d_{13}d_{32} + d_{14}d_{42} \\
\ell_2 &= d_{12}d_{21} + d_{22}^2 + d_{23}d_{32} + d_{24}d_{42} \\
\ell_3 &= d_{12}d_{31} + d_{22}d_{32} + d_{32}d_{33} + d_{34}d_{42} \\
\ell_4 &= d_{12}d_{41} + d_{22}d_{42} + d_{32}d_{43} + d_{42}d_{44}
\end{aligned}$$

Now it is easy to compute the matrices $\tilde{D} = M_C^{-1} \cdot N_C$ and $\tilde{C} = M_D^{-1} \cdot N_D$. In order to compare \tilde{C} to C and \tilde{D} to D , we have to transform from one coordinate system to the other. For instance, we can substitute the entry d_{ij} of D by the (i, j) -entry of \tilde{D} (which is a polynomial in the c_{kl}). Upon performing this substitution in \tilde{C} , the result should equal C modulo the ideal $I(\mathbb{B}_C)$. Using CoCoA (cf. [3]), it is straightforward to check that this is indeed the case.

On the side, we find that the open set $\mathbb{B}_{C, \mathcal{O}'}$ is defined as a subscheme of \mathbb{B}_C by the non-vanishing of $\det(M_C) = c_{24}c_{42}^2 - c_{22}c_{42}c_{44} - c_{22}^2$, and the open set $\mathbb{B}_{\mathcal{O}', C}$ is defined as a subscheme of $\mathbb{B}_{\mathcal{O}'}$ by the non-vanishing of $\det(M_D) = d_{31}d_{42} - d_{32}d_{41}$.

The theorem allows us to answer to above question about the relation between the matrices C_I and D_I representing I in the two \mathcal{P} -basis schemes. The explicit answer is given in the following corollary.

Corollary 2.10. *Let $I \subset P$ be an ideal such that both \mathcal{P} and \mathcal{P}' are bases modulo I . Compute $M_{C_I}, M_{D_I} \in \text{Mat}_\mu(K)$, $N_{C_I} \in \text{Mat}_{\mu\alpha'}(K)$, and $N_{D_I} \in \text{Mat}_{\mu, \alpha}(K)$ such that*

$$\mathcal{P} = \mathcal{P}' \cdot M_{D_I} \quad \mathcal{P}' = \mathcal{P} \cdot M_{C_I} \quad b\mathcal{P} = \mathcal{P}' \cdot N_{D_I} \quad b\mathcal{P}' = \mathcal{P} \cdot N_{C_I}$$

hold in P/I . Then M_{C_I} and M_{D_I} are invertible and we have $C_I = M_{D_I}^{-1} \cdot N_{D_I}$ as well as $D_I = M_{C_I}^{-1} \cdot N_{C_I}$.

Proof. It suffices to substitute the coordinate tuples representing I in $\mathbb{B}_{\mathcal{P}}$ and $\mathbb{B}_{\mathcal{P}'}$ in the matrix equalities given in part (b) of the theorem. \square

A slight change in the point of view enables us to determine the relation between the coefficients of the border bases of two ideals which differ only by a linear change of coordinates.

Proposition 2.11. *Let $I \subset P$ be an ideal such that \mathcal{P} is a basis modulo I , and let $\varphi : P \rightarrow P$ be a linear change of coordinates. Write $\varphi^{-1}(\mathcal{P}) \equiv \mathcal{P} \cdot M_\varphi \pmod{I}$ with a matrix $M_\varphi \in \text{Mat}_\mu(K)$, and write $\varphi^{-1}(b\mathcal{P}) \equiv \mathcal{P} \cdot N_\varphi \pmod{I}$ with a matrix $N_\varphi \in \text{Mat}_{\mu, \alpha}(K)$. Then the following conditions are equivalent.*

- (a) *The set \mathcal{P} is a basis modulo $\varphi(I)$.*
- (b) *The matrix M_φ is invertible.*

In this case the ideal $\varphi(I)$ is represented by $C_{\varphi(I)} = M_\varphi^{-1} \cdot N_\varphi$ in $\mathbb{B}_{\mathcal{P}}$.

Proof. By applying φ to the congruence $\varphi^{-1}(\mathcal{P}) \equiv \mathcal{P} \cdot M_\varphi \pmod{I}$, we obtain $\mathcal{P} \equiv \varphi(\mathcal{P}) \cdot M_\varphi \pmod{\varphi(I)}$. Using the fact that $\varphi(\mathcal{P})$ is a basis modulo $\varphi(I)$, we see that (a) and (b) are equivalent. Now we apply φ to the second congruence

in the proposition and get $b\mathcal{P} \equiv \varphi(\mathcal{P}) \cdot N_\varphi \pmod{\varphi(I)}$. By combining this with the previous congruence, we obtain $b\mathcal{P} \equiv \mathcal{P} \cdot M_\varphi^{-1} \cdot N_\varphi \pmod{\varphi(I)}$. Thus $\varphi(I)$ is represented by $M_\varphi^{-1} \cdot N_\varphi$ in $\mathbb{B}_\mathcal{P}$. \square

At this point we can clarify the precise meaning of the idea that a *generic* linear change of coordinates should preserve the property that I has an \mathcal{O} -basis and that we should get a flat family in this way. For this purpose, we introduce new indeterminates a_{ij} for $i = 1, \dots, n$ and $j = 0, \dots, n$, and we let $A = (a_{ij})$. The K -algebra homomorphism

$$\varphi_A : K[a_{ij}][x_1, \dots, x_n] \longrightarrow K[a_{ij}][x_1, \dots, x_n]$$

defined by $x_i \mapsto a_{i0} + a_{i1}x_1 + \dots + a_{in}x_n$ is called the **generic linear change of coordinates**. Letting $\hat{A} = (a_{ij})_{i,j=1,\dots,n}$, we see that the set of linear changes of coordinates is the open subscheme \mathbb{L} of $\mathbb{A}^{n(n+1)}$ defined by the non-vanishing of $\Delta = \det(\hat{A})$. We say that \mathbb{L} is the **scheme of linear coordinate changes**. The coordinate ring of \mathbb{L} is $K[a_{ij}]_\Delta$. Given a linear change of coordinates $\varphi : P \longrightarrow P$ such that $\varphi(x_i) = \alpha_{i0} + \alpha_{i1}x_1 + \dots + \alpha_{in}x_n$ with $\alpha_{ij} \in K$, we shall say that the matrix $\mathcal{A} = (\alpha_{ij})$, or the point in \mathbb{L} whose coordinates are the entries of \mathcal{A} , represent φ in \mathbb{L} .

Proposition 2.12. *Let $I \subset P$ be an ideal such that \mathcal{P} is a basis modulo I , let φ_A be the generic linear change of coordinates, let $\bar{I} = I \cdot K[a_{ij}]_\Delta[x_1, \dots, x_n]$, and let W be the Zariski-open subset of \mathbb{L} consisting of all points representing linear changes of coordinates $\varphi : P \longrightarrow P$ such that the matrix M_φ is invertible.*

- (a) *The open set W is non-empty.*
- (b) *Let $M_{\varphi_A} \in \text{Mat}_\mu(K[a_{ij}]_\Delta)$ be such that $\varphi_A^{-1}(\mathcal{P}) \equiv \mathcal{P} \cdot M_{\varphi_A} \pmod{\bar{I}}$, and let $\Lambda = \det(M_{\varphi_A})$. Then the coordinate ring of W is $K[a_{ij}]_{\Delta \cdot \Lambda}$.*
- (c) *The entries of the matrix $M_{\varphi_A}^{-1} \cdot N_{\varphi_A}$ are contained in $K[a_{ij}]_{\Delta \cdot \Lambda}$. Hence there exists a well-defined K -algebra homomorphism $\psi : B_\mathcal{P} \longrightarrow K[a_{ij}]_{\Delta \cdot \Lambda}$ which satisfies $\psi(c_{ij}) = (M_{\varphi_A}^{-1} \cdot N_{\varphi_A})_{i,j}$ for all i, j .*
- (d) *The map ψ induces a morphism $\Phi : W \longrightarrow \mathbb{B}_\mathcal{P}$ of affine schemes which is defined as follows. If $p \in W$ (resp. the corresponding matrix \mathcal{A}) represents a linear change of coordinates $\varphi : P \longrightarrow P$, then $\Phi(p)$ is represented by $C_{\varphi(I)} = M_\varphi^{-1} \cdot N_\varphi$.*
- (e) *The map ψ induces a flat family $\Psi : K[a_{ij}]_{\Delta \cdot \Lambda} \longrightarrow U_\mathcal{P} \otimes_{B_\mathcal{P}} K[a_{ij}]_{\Delta \cdot \Lambda}$.*

Proof. For the proof of (a), it suffices to observe that W contains the point corresponding to the identity map. To prove (b), we apply the preceding proposition. It says that, for a point $(\alpha_{ij}) \in \mathbb{L}$ representing a linear change of coordinates φ , the set \mathcal{P} is a basis modulo $\varphi(I)$ if and only if $\det(M_\varphi) = \Lambda(\alpha_{ij}) \neq 0$.

Next we show (c). The fact that the entries of $M_{\varphi_A}^{-1}$ are in $K[a_{ij}]_{\Delta \cdot \Lambda}$ follows from the definition. From $\varphi_A^{-1}(b\mathcal{P}) \equiv \mathcal{P} \cdot N_{\varphi_A} \pmod{\bar{I}}$ we see that the entries of N_{φ_A} are polynomials in the rational functions $\frac{a_{ij}}{\Delta}$. These observations show that $\psi(c_{ij}) \in K[a_{ij}]_{\Delta \cdot \Lambda}$ for all i, j .

Claim (d) follows immediately from (c), and (e) is a consequence of the flatness of the universal family (see Proposition 2.5) by applying a base change with $K[a_{ij}]_{\Delta \cdot \Lambda}$. \square

The following example illustrates the proposition.

Example 2.13. Let I be the ideal in $K[x_1, x_2]$ generated by $\{x_1^2 - 1, x_2^2 - 1\}$. Then $\mathcal{O} = (1, x_1, x_2, x_1x_2)$ is a basis modulo I . The generic linear change of coordinates φ_A is given by

$$\begin{aligned}\varphi_A(x_1) &= a_{10} + a_{11}x_1 + a_{12}x_2 \\ \varphi_A(x_2) &= a_{20} + a_{21}x_1 + a_{22}x_2\end{aligned}$$

We have $\Delta = a_{11}a_{22} - a_{12}a_{21}$. The inverse of φ_A satisfies

$$\begin{aligned}\varphi_A^{-1}(x_1) &= \frac{1}{\Delta}(a_{22}x_1 - a_{12}x_2) - b_{10} \\ \varphi_A^{-1}(x_2) &= \frac{1}{\Delta}(-a_{21}x_1 + a_{11}x_2) - b_{20}\end{aligned} \quad \text{with} \quad \begin{pmatrix} b_{10} \\ b_{20} \end{pmatrix} = \hat{A}^{-1} \cdot \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}$$

Using the fact that \mathcal{O} is a basis modulo I , we write $\varphi_A^{-1}(\mathcal{O}) = \mathcal{O} \cdot M_{\varphi_A} \pmod{\bar{I}}$ where

$$M_{\varphi_A} = \begin{pmatrix} 1 & -b_{10} & -b_{20} & b_{10}b_{20} - (a_{11}a_{12} + a_{21}a_{22})/\Delta^2 \\ 0 & a_{22}/\Delta & -a_{21}/\Delta & (a_{21}b_{10} - a_{22}b_{20})/\Delta \\ 0 & -a_{12}/\Delta & a_{11}/\Delta & (a_{12}b_{20} - a_{11}b_{10})/\Delta \\ 0 & 0 & 0 & (a_{11}a_{22} + a_{12}a_{21})/\Delta^2 \end{pmatrix}$$

Thus we have $\Lambda = \det(M_{\varphi_A}) = (a_{11}a_{22} + a_{12}a_{21})/\Delta^3$.

Now we consider a specific K -algebra homomorphism $\varphi : P \rightarrow P$ given by $x_i \mapsto \alpha_{i0} + \alpha_{i1}x_1 + \alpha_{i2}x_2$ with $\alpha_{ij} \in K$. The condition that φ is a linear change of coordinates is expressed by $\Delta(\alpha_{ij}) \neq 0$. The additional condition that M_φ is invertible is then expressed by $\Lambda(\alpha_{ij}) \neq 0$, because we have $M_\varphi = M_{\varphi_A}|_{\alpha_{ij} \mapsto \alpha_{ij}}$.

For instance, let $\varphi : K[x_1, x_2] \rightarrow K[x_1, x_2]$ be given by $\varphi(x_1) = x_1 + x_2$ and $\varphi(x_2) = x_1 - x_2$, i.e. let $\alpha_{10} = \alpha_{20} = 0$, $\alpha_{11} = \alpha_{22} = \alpha_{21} = 1$, and $\alpha_{12} = -1$. Then $\Delta(\alpha_{ij}) \neq 0$ shows that φ is invertible. Now $\Lambda(\alpha_{ij}) = 0$ implies that M_φ is not invertible. Hence \mathcal{O} is not a basis modulo $\varphi(I)$. In fact, if we perform the linear change, we see that $\varphi(I)$ is generated by $\{(x_1 - x_2)^2 - 1, (x_1 + x_2)^2 - 1\}$, and therefore by $\{x_1^2 + x_2^2 - 1, x_1x_2\}$. Since $x_1x_2 \in \varphi(I)$, it is clear that \mathcal{O} is not a basis modulo $\varphi(I)$.

The existence of a flat family of ideals defined by linear changes of coordinates distinguishes border bases from Gröbner bases in the following sense.

Corollary 2.14. *Let \mathcal{O} be an order ideal in \mathbb{T}^n , and let $I \subset P$ be an ideal which has an \mathcal{O} -border basis. Then, for a generically chosen linear change of coordinates φ , the ideal $\varphi(I)$ has again an \mathcal{O} -border basis.*

Proof. This follows from Proposition 2.11 and the fact that $W \neq \emptyset$ by the preceding proposition. \square

Notice that the property described in this corollary differs markedly from Gröbner basis theory where a generically chosen linear change of coordinates entails in general a new leading term ideal, and therefore also a new order ideal $\mathcal{O}_\sigma(I) = \mathbb{T}^n \setminus \text{LT}_\sigma(I)$. In border basis theory there is *no gin!*

Finally, we point out two particular situations in which the claims of the preceding two propositions simplify substantially.

Example 2.15. Let us consider the set of all translations, i.e. of the linear changes of coordinates with $\hat{A} = I_n$. They are all invertible and their inverses are also translations. If φ is a translation and we order the elements of \mathcal{O} in increasing degree, then M_φ is an upper triangular matrix having all entries on the main diagonal equal to 1. Therefore the matrix M_φ is invertible for every translation φ .

Hence, given an ideal $I \subset P$ such that \mathcal{O} is a basis modulo I , the order ideal \mathcal{O} is also a basis modulo $\varphi(I)$.

Example 2.16. Consider the order ideal $\mathcal{O} = \{1, x_1, \dots, x_n\}$, and let $\varphi : P \rightarrow P$ be a linear change of coordinates. We write $\varphi(x_i) = a_{i0} + a_{i1}x_1 + \dots + a_{in}x_n$ with $a_{ij} \in K$. Given an ideal I which has an \mathcal{O} -border basis, the matrix M_{φ_A} is defined by $\varphi^{-1}(\mathcal{O}) \equiv \mathcal{O} \cdot M_{\varphi_A} \pmod{I}$. Here we get

$$M_{\varphi_A} = \begin{pmatrix} 1 & -b_{10} & \cdots & -b_{n0} \\ 0 & & & \\ \vdots & & (\hat{A}^{-1})^{\text{tr}} & \\ 0 & & & \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} b_{10} \\ \vdots \\ b_{n0} \end{pmatrix} = \hat{A}^{-1} \cdot \begin{pmatrix} a_{10} \\ \vdots \\ a_{n0} \end{pmatrix}$$

Hence we have $W = \mathbb{L}$ in Proposition 2.12. In other words, if \mathcal{O} is a basis modulo I and $\varphi : P \rightarrow P$ is a linear change of coordinates, then \mathcal{O} is also a basis modulo $\varphi(I)$.

3. THE PRINCIPAL COMPONENT OF THE BORDER BASIS SCHEME

As mentioned in the introduction, our next goal is to study the principal component of the border basis scheme. Since we do not need the very general setting of the previous section, we shall concentrate on the classical border basis scheme.

In the following we continue to work over the polynomial ring $P = K[x_1, \dots, x_n]$ over a field K , we let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal of terms in \mathbb{T}^n , and we let $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$ be its border. Moreover, we denote the algebraic closure of K by \overline{K} , and we let $\overline{P} = \overline{K}[x_1, \dots, x_n]$.

Definition 3.1. For each 0-dimensional ideal $I \subset \overline{P}$ having an \mathcal{O} -border basis, let $\beta(I)$ be the corresponding point of $\mathbb{B}_{\mathcal{O}} \times_{\text{Spec}(K)} \text{Spec}(\overline{K})$. Then the closed subscheme $\mathbb{C}_{\mathcal{O}}$ of $\mathbb{B}_{\mathcal{O}}$ such that $\mathbb{C}_{\mathcal{O}} \times_{\text{Spec}(K)} \text{Spec}(\overline{K})$ is the closure of the set of all points $\beta(I_{\mathbb{X}})$, where $\mathbb{X} \subseteq \mathbb{A}^n(\overline{K})$ is a reduced scheme of length μ and $I_{\mathbb{X}} \subset \overline{P}$ is its vanishing ideal, is called the **principal component** of $\mathbb{B}_{\mathcal{O}}$.

It is known that the *radical component* of the Hilbert scheme is irreducible (see [19], 18.32). Since $\mathbb{C}_{\mathcal{O}}$ is a Zariski-open subset of the radical component, it follows that $\mathbb{C}_{\mathcal{O}}$ is an irreducible component of $\mathbb{B}_{\mathcal{O}}$, so that its name is justified. This result is also shown in Theorem 3.6 below.

As promised in the introduction, we will construct explicit equations defining $\mathbb{C}_{\mathcal{O}}$. Our method is inspired by suggestions in [6], p. 213 and [7], Sect. 2.1. We use additional indeterminates $y_j^{(i)}$ for $i = 1, \dots, \mu$ and $j = 1, \dots, n$ and we group them into tuples $\mathbf{y}^{(i)} = (y_1^{(i)}, \dots, y_n^{(i)})$. The indeterminates in $\mathbf{y}^{(i)}$ should be thought of as representing the coordinates of the i^{th} point of \mathbb{X} .

Definition 3.2. Let $Q = K[\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)}]$. We define the following polynomials in Q .

- (a) Let $\Delta_{\mathcal{O}} = \det(t_j(\mathbf{y}^{(i)}))_{i,j=1,\dots,\mu}$ where $t_j(\mathbf{y}^{(i)})$ denotes the result of the substitutions $x_k \mapsto y_k^{(i)}$ in t_j .

- (b) For $i = 1, \dots, \mu$ and $j = 1, \dots, \nu$, we let

$$\Delta_{ij} = \det(t_1(\mathbf{y}^\bullet) \mid \cdots \mid b_j(\mathbf{y}^\bullet) \mid \cdots \mid t_\mu(\mathbf{y}^\bullet))$$

Here $t_k(\mathbf{y}^\bullet)$ denotes the k^{th} column of the matrix $\Delta_{\mathcal{O}}$, i.e. the column $(t_k(\mathbf{y}^{(1)}), \dots, t_k(\mathbf{y}^{(\mu)}))^{\text{tr}}$. Thus Δ_{ij} is the determinant of the matrix where the i^{th} column of $\Delta_{\mathcal{O}}$ has been replaced by $b_j(\mathbf{y}^\bullet)$.

This definition can be motivated as follows.

Remark 3.3. Notice that $\Delta_{\mathcal{O}} \neq 0$, since each row contains different indeterminates.

- (a) In the quotient field of Q , consider the system of linear equations

$$\begin{aligned} t_1(\mathbf{y}^{(1)})z_1 + \cdots + t_\mu(\mathbf{y}^{(1)})z_\mu &= b_j(\mathbf{y}^{(1)}) \\ &\vdots \\ t_1(\mathbf{y}^{(\mu)})z_1 + \cdots + t_\mu(\mathbf{y}^{(\mu)})z_\mu &= b_j(\mathbf{y}^{(\mu)}) \end{aligned}$$

By Cramer's Rule, its solution is given by $\frac{1}{\Delta_{\mathcal{O}}} \cdot (\Delta_{1j}, \dots, \Delta_{\mu j})$.

- (b) Given a set of points $\mathbb{X} = \{\mathbf{p}_1, \dots, \mathbf{p}_\mu\}$ whose vanishing ideal $I_{\mathbb{X}}$ has an \mathcal{O} -border basis, we can substitute the coordinates p_{ij} of the points $\mathbf{p}_i = (p_{i1}, \dots, p_{in})$ for the indeterminates $y_j^{(i)}$ in the systems of linear equations above. The solutions (γ_{ij}) of the resulting systems are precisely the coefficients of the border basis $G = \{g_1, \dots, g_\nu\}$ of the ideal $I_{\mathbb{X}}$. Here we have $g_j = b_j - \sum_i \gamma_{ij} t_i$.

The main result of this subsection is that the following ring is isomorphic to the affine coordinate ring of the principal component of $\mathbb{B}_{\mathcal{O}}$.

Notation 3.4. Let $K(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)})$ be the quotient field of Q .

- (a) Let $C_{\mathcal{O}}$ be the K -subalgebra of $K(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)})$ generated by the elements $\Delta_{ij}/\Delta_{\mathcal{O}}$ with $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$.
(b) For $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$, let c_{ij} be new indeterminates. We define a surjective K -algebra homomorphism $\Phi : K[c_{ij}] \rightarrow C_{\mathcal{O}}$ by letting $\Phi(c_{ij}) = \Delta_{ij}/\Delta_{\mathcal{O}}$.

Lemma 3.5. *The defining ideal of $\mathbb{B}_{\mathcal{O}}$ is contained in the kernel of Φ . Consequently, the map Φ induces a surjective K -algebra homomorphism $B_{\mathcal{O}} \rightarrow C_{\mathcal{O}}$ and a closed immersion $\text{Spec}(C_{\mathcal{O}}) \hookrightarrow \mathbb{B}_{\mathcal{O}}$.*

Proof. The ideal $I(\mathbb{B}_{\mathcal{O}})$ defining $\mathbb{B}_{\mathcal{O}}$ is generated by the entries of the commutators $\mathcal{A}_k \mathcal{A}_\ell - \mathcal{A}_\ell \mathcal{A}_k$ of the formal multiplication matrices of the generic \mathcal{O} -border basis. Thus we have to show that the matrices $\Phi(\mathcal{A}_k)$ commute, where $\Phi(\mathcal{A}_k)$ is obtained by applying Φ to the entries of the matrix \mathcal{A}_k .

The j^{th} column of $\Phi(\mathcal{A}_k)$ is the solution of the system of linear equations

$$(t_1(\mathbf{y}^\bullet) \mid \cdots \mid t_\mu(\mathbf{y}^\bullet)) \cdot (z_1, \dots, z_\mu)^{\text{tr}} = (x_k t_j)(\mathbf{y}^\bullet)$$

Therefore we get the following equalities (*):

$$\begin{aligned} (t_1(\mathbf{y}^\bullet) \mid \cdots \mid t_\mu(\mathbf{y}^\bullet)) \cdot \Phi(\mathcal{A}_k) &= ((x_k t_1)(\mathbf{y}^\bullet) \mid \cdots \mid (x_k t_\mu)(\mathbf{y}^\bullet)) \\ &= \text{diag}(y_k^{(1)}, \dots, y_k^{(\mu)}) \cdot (t_1(\mathbf{y}^\bullet) \mid \cdots \mid t_\mu(\mathbf{y}^\bullet)) \end{aligned}$$

Since diagonal matrices commute, it follows that

$$(t_1(\mathbf{y}^\bullet) \mid \cdots \mid t_\mu(\mathbf{y}^\bullet)) \cdot \Phi(\mathcal{A}_k) \cdot \Phi(\mathcal{A}_\ell) = (t_1(\mathbf{y}^\bullet) \mid \cdots \mid t_\mu(\mathbf{y}^\bullet)) \cdot \Phi(\mathcal{A}_\ell) \cdot \Phi(\mathcal{A}_k)$$

and the fact that the matrix $(t_1(\mathbf{y}^\bullet) \mid \cdots \mid t_\mu(\mathbf{y}^\bullet))$ is invertible over the quotient field of Q implies the claim. \square

In fact, the image of the closed immersion we just found is exactly the principal component of the border basis scheme, as our next theorem shows.

Theorem 3.6. (The Coordinate Ring of the Principal Component)

Let $\Phi : K[c_{ij}] \rightarrow C_{\mathcal{O}}$ be the surjective K -algebra homomorphism defined by $\Phi(c_{ij}) = \Delta_{ij}/\Delta_{\mathcal{O}}$.

- (a) The ideal $\ker(\Phi)$ is the vanishing ideal of the principal component $\mathbb{C}_{\mathcal{O}}$ of the border basis scheme $\mathbb{B}_{\mathcal{O}}$. In particular, the principal component $\mathbb{C}_{\mathcal{O}}$ of $\mathbb{B}_{\mathcal{O}}$ is the closure of the image of the morphism $\text{Spec}(C_{\mathcal{O}}) \hookrightarrow \mathbb{B}_{\mathcal{O}}$.
- (b) The scheme $\mathbb{C}_{\mathcal{O}}$ is irreducible.

Proof. Let \overline{K} be the algebraic closure of K . Since the base change $K \subseteq \overline{K}$ is faithfully flat, it suffices to prove that the ideal $\ker(\Phi) \cdot \overline{K}[c_{ij}]$ is the vanishing ideal of the scheme $\mathbb{C}_{\mathcal{O}} \times_{\text{Spec}(K)} \text{Spec}(\overline{K})$. In other words, we may (and shall) assume that K is algebraically closed.

First we show that the map $\text{Spec}(\Phi)$ yields a bijection between the closed points of $\text{Spec}(C_{\mathcal{O}})$ and the closed points of $\mathbb{C}_{\mathcal{O}}$. A closed point (p_{ij}) of $\text{Spec}(C_{\mathcal{O}})$ corresponds to a maximal ideal $\mathfrak{m} = \langle y_j^{(i)} - p_{ij} \rangle_{i=1, \dots, \mu; j=1, \dots, n}$ of $K[\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)}]$ which does not contain $\Delta_{\mathcal{O}}$. Thus \mathfrak{m} defines also a maximal ideal in the localization $K[\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)}]_{\Delta_{\mathcal{O}}}$ and by intersecting it with $C_{\mathcal{O}}$ we obtain a maximal ideal $\mathfrak{m}_{\mathcal{O}}$ of $C_{\mathcal{O}}$.

Let us examine this maximal ideal $\mathfrak{m}_{\mathcal{O}}$ more closely. The elements $p_{ij} \in K$ define a set of points $\mathbb{X} = \{\mathbf{p}_1, \dots, \mathbf{p}_\mu\}$ where $\mathbf{p}_i = (p_{i1}, \dots, p_{in})$. The image of $\Delta_{\mathcal{O}}$ in $K[\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)}]/\mathfrak{m}$ is the determinant of $(t_j(\mathbf{p}_i))_{i,j}$. Since we assume that this determinant is non-zero, the set \mathbb{X} consists of pairwise distinct points. Moreover, it follows that the ideal $I_{\mathbb{X}}$ has an \mathcal{O} -border basis.

The systems of linear equations $(t_1(\mathbf{p}_i) \mid \cdots \mid t_\mu(\mathbf{p}_i)) \cdot (z_1, \dots, z_\mu)^{\text{tr}} = (b_j(\mathbf{p}_i))$ have unique solutions $(\gamma_{1j}, \dots, \gamma_{\mu j}) \in K^\mu$. Then the corresponding \mathcal{O} -border pre-basis $G = \{g_1, \dots, g_\nu\}$ with $g_j = b_j - \sum_{i=1}^\mu \gamma_{ij} t_i$ is the \mathcal{O} -border basis of $I_{\mathbb{X}}$ and the maximal ideal $\langle c_{ij} - \gamma_{ij} \rangle_{i,j}$ of $K[c_{ij}]$ defines a closed point of $\mathbb{C}_{\mathcal{O}}$. By Remark 3.3.b, the maximal ideal $\langle c_{ij} - \gamma_{ij} \rangle_{i,j}$ is precisely the preimage of \mathfrak{m} under Φ .

Conversely, let us start with a closed point (γ_{ij}) of $\mathbb{C}_{\mathcal{O}}$ corresponding to the \mathcal{O} -border basis of a radical ideal I . Since the base field is algebraically closed, the ideal I is the vanishing ideal of a set of μ points $\mathbb{X} = \{(p_{i1}, \dots, p_{in}) \mid i = 1, \dots, \mu\}$ in \mathbb{A}^n . From what we just showed it follows that the maximal ideal $\langle c_{ij} - \gamma_{ij} \rangle_{i,j}$ is the preimage of the maximal ideal $\langle y_j^{(i)} - p_{ij} \rangle_{i=1, \dots, \mu; j=1, \dots, n}$ under Φ .

Next we let $R = K[\Delta_{ij} \mid i \in \{1, \dots, \mu\}, j \in \{1, \dots, \nu\}]_{\Delta_{\mathcal{O}}}$ and consider the canonical injective K -algebra homomorphism $C_{\mathcal{O}} \hookrightarrow R$. The intersection of the maximal ideals of R is (0) . The preimages of these maximal ideals in $C_{\mathcal{O}}$ are exactly the maximal ideals $\mathfrak{m}_{\mathcal{O}}$ constructed above. Hence the intersection of the maximal ideals $\mathfrak{m}_{\mathcal{O}}$ is (0) . From what we have shown, it follows that the intersection of the maximal ideals $\langle c_{ij} - \gamma_{ij} \rangle_{i,j}$ of $K[c_{ij}]$ is the kernel of Φ .

On the other hand, the general form of Hilbert's Nullstellensatz (see [14], 2.6.17) implies that the intersection of the maximal ideals $\langle c_{ij} - \gamma_{ij} \rangle_{i,j}$ of $K[c_{ij}]$ is the

vanishing ideal of the closure of the corresponding set of points, i.e. the vanishing ideal of the principal component $\mathbb{C}_{\mathcal{O}}$ of $\mathbb{B}_{\mathcal{O}}$.

To prove (b), we note that the ring $C_{\mathcal{O}}$ is a subalgebra of $K(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)})$. Hence it is an integral domain. Therefore the ideal $\ker(\Phi)$ is a prime ideal. \square

In view of the preceding theorem it will prove useful to study the K -algebra $C_{\mathcal{O}}$ in more detail. Our next proposition shows that it contains the following elements.

Notation 3.7. Let $L = \{s_1, \dots, s_{\mu}\}$ be a set of μ distinct terms contained in \mathbb{T}^n . Then we set $\Delta_L = \det(s_1(\mathbf{y}^{(i)}) \mid \dots \mid s_{\mu}(\mathbf{y}^{(i)})) \in Q$.

Proposition 3.8. For every $L = \{s_1, \dots, s_{\mu}\} \subset \mathbb{T}^n$, we have $\Delta_L / \Delta_{\mathcal{O}} \in C_{\mathcal{O}}$.

Proof. If $L = \mathcal{O}$, we have $\Delta_L / \Delta_{\mathcal{O}} = 1 \in C_{\mathcal{O}}$. Next we show that $\Delta_{L_j} / \Delta_{\mathcal{O}} \in C_{\mathcal{O}}$ for $L_j = (t_1, \dots, t_{j-1}, s, t_{j+1}, \dots, t_{\mu})$ with $j \in \{1, \dots, \mu\}$ and a term $s \in \mathbb{T}^n \setminus \mathcal{O}$. We write $s = t' b_j$ with $t' \in \mathbb{T}^n$, $j \in \{1, \dots, \nu\}$ and we prove the claim by induction on $\deg(t')$.

If $\deg(t') = 0$, the term s is a border term and Δ_L is one of the elements Δ_{ij} . Now let $\deg(t') > 0$ and write $t' = x_k t''$ with $k \in \{1, \dots, n\}$ and $t'' \in \mathbb{T}^n$. By Cramer's rule, we have

$$\begin{aligned} (t_1(\mathbf{y}^{\bullet}) \mid \dots \mid t_{\mu}(\mathbf{y}^{\bullet})) \cdot (\Delta_{L_1} / \Delta_{\mathcal{O}}, \dots, \Delta_{L_{\mu}} / \Delta_{\mathcal{O}})^{\text{tr}} &= ((x_k t'' b_j)(\mathbf{y}^{\bullet})) \\ &= \text{diag}(y_k^{(1)}, \dots, y_k^{(\mu)}) \cdot ((t'' b_j)(\mathbf{y}^{\bullet})) \end{aligned}$$

The inductive hypothesis implies that there are elements $\tilde{\Delta}_1 / \Delta_{\mathcal{O}}, \dots, \tilde{\Delta}_{\mu} / \Delta_{\mathcal{O}} \in C_{\mathcal{O}}$ such that

$$(t_1(\mathbf{y}^{\bullet}) \mid \dots \mid t_{\mu}(\mathbf{y}^{\bullet})) \cdot (\tilde{\Delta}_1 / \Delta_{\mathcal{O}}, \dots, \tilde{\Delta}_{\mu} / \Delta_{\mathcal{O}})^{\text{tr}} = ((t'' b_j)(\mathbf{y}^{\bullet}))$$

Using the equality (*) from the proof of Lemma 3.5, we get

$$\begin{aligned} (t_1(\mathbf{y}^{\bullet}) \mid \dots \mid t_{\mu}(\mathbf{y}^{\bullet})) \cdot (\Delta_{L_1} / \Delta_{\mathcal{O}}, \dots, \Delta_{L_{\mu}} / \Delta_{\mathcal{O}})^{\text{tr}} &= \\ &= \text{diag}(y_k^{(1)}, \dots, y_k^{(\mu)}) \cdot (t_1(\mathbf{y}^{\bullet}) \mid \dots \mid t_{\mu}(\mathbf{y}^{\bullet})) \cdot (\tilde{\Delta}_1 / \Delta_{\mathcal{O}}, \dots, \tilde{\Delta}_{\mu} / \Delta_{\mathcal{O}})^{\text{tr}} \\ &= (t_1(\mathbf{y}^{\bullet}) \mid \dots \mid t_{\mu}(\mathbf{y}^{\bullet})) \cdot \tilde{\Phi}(\mathcal{A}_k) \cdot (\tilde{\Delta}_1 / \Delta_{\mathcal{O}}, \dots, \tilde{\Delta}_{\mu} / \Delta_{\mathcal{O}})^{\text{tr}} \end{aligned}$$

At this point we note that $(t_1(\mathbf{y}^{\bullet}) \mid \dots \mid t_{\mu}(\mathbf{y}^{\bullet}))$ is an invertible matrix over the field $K(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)})$. It follows that the tuple $(\Delta_{L_1} / \Delta_{\mathcal{O}}, \dots, \Delta_{L_{\mu}} / \Delta_{\mathcal{O}})$ is contained in $(C_{\mathcal{O}})^{\mu}$.

Finally we turn to the general case. Let $L = \{t_{i_1}, \dots, t_{i_k}, s_1, \dots, s_{\mu-k}\}$ with $i_1, \dots, i_k \in \{1, \dots, \mu\}$ and $s_j \in \mathbb{T}^n \setminus \mathcal{O}$. Clearly, we may assume that the indices i_1, \dots, i_k are pairwise distinct. We proceed by downward induction on k . The case $k = \mu - 1$ has been treated above. For the induction step, let $\{i_{k+1}, \dots, i_{\mu-k}\} = \{1, \dots, \mu\} \setminus \{i_1, \dots, i_k\}$. Now the claim follows from the Plücker relation

$$\begin{aligned} \Delta_L \cdot \Delta_{\mathcal{O}} &= \det(t_{i_1}(\mathbf{y}^{\bullet}) \mid \dots \mid t_{i_k}(\mathbf{y}^{\bullet}) \mid s_1(\mathbf{y}^{\bullet}) \mid \dots \mid s_{\mu-k}(\mathbf{y}^{\bullet})) \\ &\quad \cdot \det(t_{i_1}(\mathbf{y}^{\bullet}) \mid \dots \mid t_{i_{\mu}}(\mathbf{y}^{\bullet})) \\ &= \sum_{j=1}^{\mu-k} \pm \det(t_{i_1}(\mathbf{y}^{\bullet}) \mid \dots \mid t_{i_{k+1}}(\mathbf{y}^{\bullet}) \mid s_1(\mathbf{y}^{\bullet}) \mid \dots \mid \widehat{s_j(\mathbf{y}^{\bullet})} \mid \dots \mid s_{\mu-k}(\mathbf{y}^{\bullet})) \\ &\quad \cdot \det(t_{i_1} \mid \dots \mid \widehat{t_{i_k}(\mathbf{y}^{\bullet})} \mid \dots \mid t_{i_{\mu-k}}(\mathbf{y}^{\bullet}) \mid s_j(\mathbf{y}^{\bullet})) \end{aligned}$$

and the inductive hypothesis. \square

In other words, this proposition says that $C_{\mathcal{O}} = K[\Delta_L/\Delta_{\mathcal{O}} \mid L \subset \mathbb{T}^n, \#L = \mu]$. Therefore the ring $C_{\mathcal{O}}$ agrees with the one mentioned in [6] and [7]. Restricting the number of algebra generators has an obvious advantage: we can now write down an algorithm for computing the defining equations of the principal component. This makes it possible to check effectively whether a given border basis scheme is irreducible.

Proposition 3.9. *Let $\mathcal{O} = \{t_1, \dots, t_{\mu}\}$ be an order ideal in \mathbb{T}^n and let c_{ij} be further indeterminates, where $i = 1, \dots, \mu$ and $j = 1, \dots, \nu$. The following instructions define an algorithm which computes a system of generators of the defining ideal in $K[c_{ij}]$ of the principal component $\mathbb{C}_{\mathcal{O}}$ of the border basis scheme.*

- (1) *Form the polynomial ring $Q = K[\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)}]$ and compute the elements $\Delta_{\mathcal{O}} = \det(t_j(\mathbf{y}^{(i)}))$ and $\Delta_{ij} = \det(t_1(\mathbf{y}^{(i)}) \mid \dots \mid b_j(\mathbf{y}^{(i)}) \mid \dots \mid t_{\mu}(\mathbf{y}^{(i)}))$ for $i = 1, \dots, \mu$ and $j = 1, \dots, \nu$.*
- (2) *Form the polynomial ring $K[c_{ij}, z]$ where z is a new indeterminate. Let I be the ideal generated by $\Delta_{\mathcal{O}} z - 1$ and the set of all $\Delta_{\mathcal{O}} c_{ij} - \Delta_{ij}$ such that $i = 1, \dots, \mu$ and $j = 1, \dots, \nu$.*
- (3) *Compute a set of generators $\{F_1, \dots, F_r\}$ of the elimination ideal $I \cap K[c_{ij}]$ and return it.*

Proof. This is a special case of a classical algorithm which computes explicit representations of finitely generated subalgebras of function fields (see for instance [15], Tutorial 41). \square

Proposition 3.10. *Let $W \in \text{Mat}_{m,n}(\mathbb{Z})$, and let P be graded by W . We equip $K[c_{ij}]$ with a \mathbb{Z}^m -grading given by a matrix \overline{W} such that $\deg_{\overline{W}}(c_{ij}) = \deg_W(b_j) - \deg_W(t_i)$. Then the elimination ideal $I \cap K[c_{ij}]$ of the preceding proposition is homogeneous with respect to the grading given by \overline{W} .*

Proof. First we introduce a \mathbb{Z}^m -grading on $Q = K[\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)}]$ given by a matrix \widetilde{W} by letting $\deg_{\widetilde{W}}(y_j^{(i)}) = \deg_W(x_i)$. Thus the elements of the j^{th} column of the matrix $t_j(\mathbf{y}^{(i)})$ are homogeneous of degree $\deg_W(t_j)$. Hence $\Delta_{\mathcal{O}}$ is homogeneous of degree $\deg_W(t_1 \cdots t_{\mu})$. Similarly, we see that Δ_{ij} is homogeneous of degree $\deg_W(t_1 \cdots t_{\mu}) - \deg_W(t_i) + \deg_W(b_j)$. This shows that if we define $\deg_{\overline{W}}(c_{ij}) = \deg_W(b_j) - \deg_W(t_i)$ and $\deg_{\overline{W}}(z) = -\deg_W(t_1 \cdots t_{\mu})$, then I is a homogeneous ideal in $K[c_{ij}, z]$. Consequently, also $I \cap K[c_{ij}]$ is a homogeneous ideal in $K[c_{ij}]$ with respect to the grading given by \overline{W} . \square

This result is in accordance with the fact that the ideal $I(\mathbb{B}_{\mathcal{O}})$ is homogeneous with respect to the same grading.

Remark 3.11. Suppose that the order ideal \mathcal{O} is a σ -cornercut with respect to some term ordering σ . This implies that there exists a system of positive weights for x_1, \dots, x_n such that $\deg_W(b_j) > \deg_W(t_i)$ for $i = 1, \dots, \mu$ and $j = 1, \dots, \nu$. By the proposition, the ideal $I \cap K[c_{ij}]$ is homogeneous with respect to a positive grading on $K[c_{ij}]$. This observation agrees with the fact that, in this case, the border basis scheme $\mathbb{B}_{\mathcal{O}}$ and the Gröbner basis scheme $\mathbb{G}_{\mathcal{O}, \sigma}$ are isomorphic (see [20], Proposition 3.11), and the latter can be seen as a weighted projective scheme (see [20], Theorem 2.8).

4. LOCAL PARAMETERS AT THE RADICAL POINTS

For a radical ideal I having an \mathcal{O} -border basis, we shall call the corresponding point of $\mathbb{C}_{\mathcal{O}}$ a **radical point**. In the following, we want to construct explicit local parameters for $\mathbb{C}_{\mathcal{O}}$ near its radical points. As a consequence, we shall recover the well-known facts that $\mathbb{C}_{\mathcal{O}}$ is smooth of dimension μn at these points, and that it is a rational variety.

The idea of our construction is to use the complete intersection representation of a radical ideal I having an \mathcal{O} -border basis which is provided by the Shape Lemma (cf. [14], Theorem 3.7.25). Recall that a 0-dimensional ideal I is said to be in **normal ℓ -position** for some $\ell \in P_1$ if we have $\ell(p) \neq \ell(q)$ for distinct points $p, q \in \text{Supp}(\mathcal{Z}(I))$. Here the zero scheme $\mathcal{Z}(I)$ of I is defined over the algebraic closure \overline{K} of K .

Proposition 4.1. *Let I be a 0-dimensional radical ideal in P which has an \mathcal{O} -border basis. Assume that K has at least $\binom{\mu}{2} + 1$ elements.*

- (a) *It is possible to choose $\ell \in P_1$ such that $\mathcal{Z}(I)$ is in normal ℓ -position.*
- (b) *Write $\ell = \ell_1 x_1 + \cdots + \ell_n x_n \in P_1$ with $\ell_1, \dots, \ell_n \in K$ and assume that $\ell_n \neq 0$. Then we have $P/I \cong K[\ell]$ and the minimal polynomial of $\bar{\ell}$ in P/I is of the form $\chi(\ell) = \ell^\mu - \lambda_\mu \ell^{\mu-1} - \cdots - \lambda_2 \ell - \lambda_1$ where $\lambda_i \in K$.*
- (c) *The ideal I has a representation*

$$I = (\chi(\ell), x_1 - f_1(\ell), \dots, x_{n-1} - f_{n-1}(\ell))$$

where the polynomials $f_i(\ell) \in K[\ell]$ have degree $\leq \mu - 1$.

Proof. For claim (a), see [14], Proposition 3.7.22. Claim (b) follows from [14], Theorem 3.7.23, and (c) is the version of the Shape Lemma given in [14], Theorem 3.7.25. \square

Using the terminology of Section 2, the set $\mathcal{P} = \{1, \ell, \dots, \ell^{\mu-1}\}$ is a pseudo order ideal because it is the image of the order ideal $\{1, x_n, \dots, x_n^{\mu-1}\}$ under the linear change of coordinates $\varphi : P \rightarrow P$ given by $\varphi(x_i) = x_i$ for $i = 1, \dots, n-1$ and $\varphi(x_n) = \ell$. Next we define a grading by $\deg_W(x_i) = \mu$ for $i = 1, \dots, n-1$ and $\deg_W(x_n) = 1$, and we choose a term ordering σ which is compatible with this grading. Then the set $\{1, x_n, \dots, x_n^{\mu-1}\}$ is a σ -cornercut and its border is the set $\{x_1, \dots, x_{n-1}, x_n^\mu\}$. Hence the set $b\mathcal{P} = \{x_1, \dots, x_{n-1}, \ell^\mu\}$ is the pseudo border of \mathcal{P} .

Another way of stating the last claim of the proposition is to say that the set $H = \{\chi(\ell), x_1 - f_1(\ell), \dots, x_{n-1} - f_{n-1}(\ell)\}$ is a pseudo \mathcal{P} -border basis of I . Pseudo border bases of this shape are parametrized by $n\mu$ coefficients, namely the coefficients $\lambda_1, \dots, \lambda_\mu$ of χ and the $(n-1)\mu$ coefficients of f_1, \dots, f_{n-1} . As $n\mu$ is the dimension of $\mathbb{C}_{\mathcal{O}}$ at the point corresponding to I (see [19], 18.32), we shall now use the base change technique of Section 2 to parametrize the principal component locally as follows. A similar result is shown in [12] using a different technique.

Theorem 4.2. *Let I be a 0-dimensional radical ideal in P which has an \mathcal{O} -border basis. Suppose that there exist a linear form $\ell = \ell_1 x_1 + \cdots + \ell_n x_n$ with $\ell_i \in K$ such that $\ell_n \neq 0$ and polynomials $\chi(\ell), f_i(\ell) \in K[\ell]$ such that*

$$I = (\chi(\ell), x_1 - f_1(\ell), \dots, x_{n-1} - f_{n-1}(\ell))$$

and such that $\chi(\ell) = \ell^\mu - \lambda_\mu \ell^{\mu-1} - \dots - \lambda_2 \ell - \lambda_1$ and $\deg(f_i(\ell)) < \mu$. For every tuple $d = (d_{ij}) \in K^{n\mu}$, we define $\tilde{\chi}(\ell) = \ell^\mu - \sum_{i=1}^\mu (\lambda_i + d_{ni}) \ell^{i-1}$ and $\tilde{f}_i(\ell) = f_i(\ell) + \sum_{j=1}^\mu d_{ij} \ell^{j-1}$ where $i \in \{1, \dots, n-1\}$. Then we let

$$I_d = (\tilde{\chi}(\ell), x_1 - \tilde{f}_1(\ell), \dots, x_{n-1} - \tilde{f}_{n-1}(\ell))$$

- (a) For all tuples d in a non-empty Zariski open neighborhood U of $(0, \dots, 0)$ in $K^{n\mu}$, the ideal I_d has an \mathcal{O} -border basis which can be computed by viewing $H_d = \{\tilde{\chi}(\ell), x_1 - \tilde{f}_1(\ell), \dots, x_{n-1} - \tilde{f}_{n-1}(\ell)\}$ as a pseudo \mathcal{P} -border basis of I_d with respect to $\mathcal{P} = \{1, \ell, \dots, \ell^{\mu-1}\}$ and by applying Corollary 2.10 to it.
- (b) The morphism $\Gamma : U \rightarrow \mathbb{C}_{\mathcal{O}}$ given by $d \mapsto I_d$ yields an isomorphism between the open set U in $K^{n\mu}$ and the open set $\mathbb{B}_{\mathcal{O}, \mathcal{P}} \cap \mathbb{C}_{\mathcal{O}}$ in $\mathbb{C}_{\mathcal{O}}$. In particular, the variety $\mathbb{C}_{\mathcal{O}}$ is rational, and it is smooth of dimension $n\mu$ at its radical points.

Proof. Let us consider the pseudo order ideal $\mathcal{P} = \{1, \ell, \dots, \ell^{\mu-1}\}$ as the image of the cornercut $\{1, x_1, \dots, x_n^{\mu-1}\}$ under a linear change of coordinates. Then its pseudo border is $b\mathcal{P} = \{x_1, \dots, x_{n-1}, \ell^\mu\}$, and $H = \{\chi(\ell), x_1 - f_1(\ell), \dots, x_{n-1} - f_{n-1}(\ell)\}$ is the pseudo \mathcal{P} -border basis of I .

First we prove (a). The ideal I has both an \mathcal{O} -border basis and a pseudo \mathcal{P} -border basis. For every $d \in K^{n\mu}$, the set H_d is a pseudo \mathcal{P} -border basis of I_d . By Theorem 2.7.a, there is a non-empty Zariski open neighbourhood U of $(0, \dots, 0)$ in $K^{n\mu}$ such that I_d has an \mathcal{O} -border basis for all $d \in U$. It is the open set defined by $\det(M_D) \neq 0$ where $M_D \in \text{Mat}_\mu(K[d_{ij}])$ is the matrix such that $\mathcal{O} \equiv \mathcal{P} \cdot M_D \pmod{I_d}$ where we view the elements d_{ij} as indeterminates. For all $d \in U$, we can use Corollary 2.10 to compute the \mathcal{O} -border basis of I_d . For this purpose, we have to calculate matrices $M_d \in \text{Mat}_\mu(K)$ and $N_d \in \text{Mat}_{\mu, \nu}(K)$ such that $\mathcal{O} = \mathcal{P} \cdot M_d$ and $\partial\mathcal{O} = \mathcal{P} \cdot N_d$ hold in P/I_d . Then the matrix representing the ideal I_d in $\mathbb{B}_{\mathcal{O}}$ is given by $C_{I_d} = (M_d)^{-1} \cdot N_d$.

It remains to prove (b). The morphism Γ sends a tuple $d = (d_{ij})$ to the point represented by the matrix $C_{I_d} = (M_d)^{-1} \cdot N_d$. We claim that it maps the open set $U = K^{n\mu} \setminus \mathcal{Z}(\det(M_C))$ isomorphically to the open set $V = \mathbb{B}_{\mathcal{O}, \mathcal{P}} \cap \mathbb{C}_{\mathcal{O}}$ in $\mathbb{C}_{\mathcal{O}}$. Given a matrix C_J representing a point in V , we know that the corresponding ideal J has both an \mathcal{O} -border basis and a pseudo \mathcal{P} -border basis. Using Corollary 2.10 again, we can compute a tuple $d \in U$ such that $J = I_d$, and this yields a morphism from V to U which inverts Γ . \square

It is well-known that $\mathbb{B}_{\mathcal{O}} = \mathbb{C}_{\mathcal{O}}$ is non-singular for the case of $n = 2$ indeterminates and that the following example shows that $\mathbb{C}_{\mathcal{O}}$ is not always non-singular at its non-radical points.

Example 4.3. Let $P = \mathbb{Q}[x, y, z]$, and let $\mathcal{O} = \{1, x, y, z\}$. Then $\mathbb{B}_{\mathcal{O}} = \mathbb{C}_{\mathcal{O}}$ is a scheme of dimension 12 in \mathbb{A}^{24} . We compute $I(\mathbb{B}_{\mathcal{O}})$ and see that we can project $\mathbb{B}_{\mathcal{O}}$ isomorphically to an 18-dimensional affine space by eliminating c_{11}, \dots, c_{16} (cf. [16]). The result is a variety $\pi(\mathbb{B}_{\mathcal{O}}) \subset \mathbb{A}^{18}$ whose vanishing ideal is generated by 15 homogeneous polynomials of degree 2. Hence its vertex $(0, \dots, 0)$ is singular. The corresponding ideal is the border term ideal $\langle \partial\mathcal{O} \rangle = \langle x^2, xy, xz, y^2, yz, z^2 \rangle$.

In [7], Sect. 2, it is explained that $\mathbb{C}_{\mathcal{O}}$ can be realized as the blowup of the Chow variety $\text{Spec}(K[\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(\mu)}]^{S_\mu})$ at an explicitly given ideal. A different

construction for $\mathbb{C}_{\mathcal{O}}$, permitting similar conclusions, is contained in [4]. Although it is in principle possible from these constructions to obtain local parameters for $\mathbb{C}_{\mathcal{O}}$ at its radical points, we believe that our construction is more elementary and explicit.

Let us compare the results of Section 2 to the preceding theorem. Given an arbitrary K -rational point of $\mathbb{B}_{\mathcal{O}}$, i.e. an ideal I having an \mathcal{O} -border basis, we can use linear changes of coordinates as in Proposition 2.12 to construct a flat family of ideals whose base space is an open subset of an $n(n+1)$ -dimensional affine space and whose special fiber is I . However, this is in general much smaller than the local dimension of $\mathbb{B}_{\mathcal{O}}$ at the point representing I . If I is reduced, Theorem 4.2 allows us to do much better: we construct a flat family over a $n\mu$ -dimensional base space, and this is precisely the local dimension of $\mathbb{B}_{\mathcal{O}}$ at the point representing I .

An application of the preceding theorem is the possibility to connect two arbitrary radical ideals having \mathcal{O} -border bases via a sequence of two explicit flat families.

Remark 4.4. Let K be an infinite field, let $P = K[x_1, \dots, x_n]$, let $I, I' \subset P$ be two radical ideals which have \mathcal{O} -border bases, and let $c_I, c_{I'}$ be the points in $\mathbb{C}_{\mathcal{O}}$ representing them.

For a generically chosen $\ell \in P_1$, the hypotheses of the theorem are satisfied with respect to I and I' (see [14], Prop. 3.7.22). By part (c) of the theorem, there exist an open neighborhood U of c_I in $\mathbb{C}_{\mathcal{O}}$ and an open neighborhood U' of $c_{I'}$ in $\mathbb{C}_{\mathcal{O}}$ such that the restriction of the universal flat family to U and U' is given by explicitly defined morphisms.

Since $\mathbb{C}_{\mathcal{O}}$ is irreducible, there exists a point $c_J \in U \cap U'$ representing a radical ideal J . Both U and U' are isomorphic to open subsets of $\mathbb{A}^{n\mu}$. Our task is to find an explicit flat family connecting c_I and c_J . In an analogous way, we can then connect $c_{I'}$ and c_J .

The points $p_I = \Gamma^{-1}(c_I)$ and $p_J = \Gamma^{-1}(c_J)$ are contained in an open subset U of the affine space $\mathbb{A}^{n\mu}$. By Theorem 2.7, we have an explicit polynomial F whose non-vanishing defines this open set. Thus we can connect the points p_I and p_J by a line L and get a K -algebra homomorphism $K[d_{ij}]_F \rightarrow K[z]_f$ which represents the inclusion $(L \cap U) \subseteq U$. Here $f \in K[z] \setminus \{0\}$ defines the points in $L \setminus U$.

After applying Γ , we get an explicit punctured rational curve $\Psi : C_{\mathcal{O}} \rightarrow K[z]_f$ which connects c_I to c_J in $\mathbb{B}_{\mathcal{O}, P} \cap \mathbb{C}_{\mathcal{O}}$. Then, by restricting the universal flat family $\Phi : B_{\mathcal{O}} \rightarrow U_{\mathcal{O}}$ to this punctured rational curve, we find a flat family deforming P/I to P/J .

Another application of Theorem 4.2 is the possibility to connect an arbitrary radical point of $\mathbb{B}_{\mathcal{O}}$ to the **monomial point** $o = (0, \dots, 0)$ representing the monomial ideal $\langle b_1, \dots, b_{\nu} \rangle$ via explicitly defined flat families. For this application we need one further ingredient, namely distractions, which we are now going to recall from [15], Section 6.2. To simplify the discussion, let us assume that the field K has sufficiently many elements.

Definition 4.5. For $i = 1, \dots, n$, let $\pi_i = (c_{i1}, c_{i2}, \dots)$ be tuples consisting of sufficiently many pairwise distinct elements of K .

- (1) For a term $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, the polynomial $D_{\pi}(t) = \prod_{i=1}^n \prod_{j=1}^{\alpha_i} (x_i - c_{ij})$ is called the **distraction** of t with respect to $\pi = (\pi_1, \dots, \pi_n)$.
- (2) Let $I = \langle b_1, \dots, b_{\nu} \rangle$ be the border term ideal of \mathcal{O} , and let $c\mathcal{O} = \{c_1, \dots, c_r\}$ be the corner set of \mathcal{O} , i.e. the set of minimal monomial generators of I .

Then the ideal $D_\pi(I) = \langle D_\pi(c_1), \dots, D_\pi(c_r) \rangle$ is called the **distraction** of I with respect to π .

The following properties of the distraction of the border term ideal are shown in [15], Theorem 6.2.12.

Proposition 4.6. *Let $\pi = (\pi_1, \dots, \pi_n)$ be chosen as in the preceding definition, and let $c\mathcal{O} = \{c_1, \dots, c_r\}$ be the corner set of the border term ideal I of \mathcal{O} .*

- (1) *The distraction $D_\pi(I)$ is a radical ideal.*
- (2) *For every term ordering σ , the set $\{D_\pi(c_1), \dots, D_\pi(c_r)\}$ is the reduced σ -Gröbner basis of $D_\pi(I)$. In particular, we have $\text{LT}_\sigma(D_\pi(I)) = I$ and $\mathbb{T}^n \setminus \text{LT}_\sigma(I) = \mathcal{O}$.*
- (3) *The ideal $D_\pi(I)$ is 0-dimensional and has an \mathcal{O} -border basis.*

Now we are ready to connect any radical point of $\mathbb{B}_\mathcal{O}$ to the monomial point via three explicit flat families.

Remark 4.7. Let K be an infinite field. Suppose we are given a reduced 0-dimensional ideal I in P which has an \mathcal{O} -border bases. To find explicit flat families connecting P/I to $P/\langle b_1, \dots, b_\nu \rangle$, we proceed as follows.

- (1) For $i = 1, \dots, n$, choose tuples π_i of sufficiently many distinct elements of K . For every $\lambda \in K$, let $\lambda\pi = (\lambda\pi_1, \dots, \lambda\pi_n)$. Then form the family $\Pi : \mathbb{A}^1 \rightarrow \mathbb{C}_\mathcal{O}$ defined by $\lambda \mapsto D_{\lambda\pi}(\langle b_1, \dots, b_\nu \rangle)$. By Proposition 4.6, there exists a family of polynomials which yields a Gröbner basis of each fiber of this family. Hence Π is a flat deformation connecting the border term ideal to the radical ideal $J = D_\pi(\langle b_1, \dots, b_\nu \rangle)$.
- (2) Now use Remark 4.4 to find two explicit flat families connecting P/J to P/I .

Notice that by using the method of the previous remark we may not always find a non-punctured rational curve in $\mathbb{C}_\mathcal{O}$ connecting c_I to the monomial point o , although such a curve might exist.

We end this section with some examples which illustrate the construction of the explicit flat families in Remarks 4.4 and 4.7. In the first one we connect the points c_I and c_J corresponding to two radical ideals by a rational curve in $\mathbb{B}_\mathcal{O}$, but one point of the rational curve is not contained in the open set $\Gamma(U) = \mathbb{B}_{\mathcal{O}, \mathcal{P}} \cap \mathbb{C}_\mathcal{O}$ of Theorem 4.2.b.

Example 4.8. Let $K = \mathbb{Q}$, let $P = \mathbb{Q}[x, y]$, and let $\mathcal{O} = \{1, x, y, xy\}$. The vanishing ideals of the two point sets $\mathbb{X} = \{(0, 0), (0, 1), (1, -1), (1, 2)\}$ and $\mathbb{Y} = \{(0, 0), (0, 1), (1, -1), (-1, 2)\}$ both have \mathcal{O} -border bases, namely

$$\begin{aligned} \mathcal{I}(\mathbb{X}) &= (y^2 - 2x - y, x^2 - x, xy^2 - 2x - xy, x^2y - xy) \\ \mathcal{I}(\mathbb{Y}) &= (y^2 - \tfrac{2}{3}x + y + \tfrac{4}{3}xy, x^2 - \tfrac{1}{3}x + \tfrac{2}{3}xy, xy^2 - 2x - xy, x^2y + \tfrac{4}{3}x + \tfrac{1}{3}xy) \end{aligned}$$

Now we use the explicit description of $\mathbb{B}_\mathcal{O}$ worked out in [16], Example 3.8. It provides an isomorphism $\Gamma : \mathbb{A}^8 \rightarrow \mathbb{B}_\mathcal{O}$ which corresponds to the \mathbb{Q} -algebra homomorphism

$$B_\mathcal{O} \rightarrow \mathbb{Q}[c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}]$$

given by $(c_{11}, c_{12}, \dots, c_{44}) \mapsto (c_{21}, c_{23}, \dots, c_{44})$. Under this isomorphism, the point representing $\mathcal{I}(\mathbb{X})$ corresponds to $(c_{21}, \dots, c_{44}) = (2, 2, 0, 0, 0, 0, 1, 1) = p_1$, and the point representing $\mathcal{I}(\mathbb{Y})$ corresponds to $p_2 = (\frac{2}{3}, 2, 0, 0, -\frac{4}{3}, -\frac{2}{3}, 1, -\frac{1}{3})$.

To connect p_1 and p_2 , we use the line $L = \{(-1 + 2a)p_1 + (\frac{1}{2} - \frac{1}{2}a)p_2 | a \in \mathbb{Q}\}$. In this way we get the point p_1 for $a = 1$ and the point p_2 for $a = -1$. For the corresponding family of ideals, we use

$$c_{21} = \frac{2}{3}a + \frac{4}{3}, c_{23} = 2, c_{32} = c_{34} = 0, c_{41} = \frac{2}{3}a - \frac{2}{3}, c_{42} = \frac{1}{3}a - \frac{1}{3}, c_{43} = 1, c_{44} = \frac{2}{3}a + \frac{1}{3}$$

This yields the family of ideals whose border bases are represented parametrically by

$$\begin{aligned} G_a = & \{y^2 - (-\frac{2}{3}a^2 + \frac{2}{3}) - (\frac{2}{3}a + \frac{4}{3})x - (-\frac{2}{3}a^2 + \frac{5}{3})y - (\frac{2}{3}a - \frac{2}{3})xy, \\ & x^2 - (\frac{1}{3}a + \frac{2}{3})x - (\frac{1}{3}a - \frac{1}{3})xy, \\ & xy^2 - 2x - xy, \\ & x^2y - (\frac{2}{3}a - \frac{2}{3})x - (\frac{2}{3}a + \frac{1}{3})xy\} \end{aligned}$$

The ideal I_a generated by G_a satisfies

$$I_0 = (x - 1, y + 1) \cap (x, y + \frac{1}{3}) \cap (x^2, y - 2)$$

in the case $a = 0$ and

$$I_a = (x - 1, y + 1) \cap (x - a, y - 2) \cap (x, \frac{3}{2}y^2 + (a^2 - \frac{5}{2})y + (a^2 - 1))$$

for $a \neq 0$. Thus the ideal I_0 is not reduced, but all other ideals of the family are.

Geometrically, this can be explained as follows. No set of points in the family can have three points on the line $x = 0$, since then the polynomial $xy + x$ vanishes on all four points and there is no \mathcal{O} -border basis. Therefore the point $(0, 1)$ “moves up” and helps the point $(a, 2)$ to get across this line by forming a non-reduced scheme.

The punctured rational curves constructed in Remark 4.4 may sometimes be restrictions of (non-punctured) rational curves on the Hilbert scheme whose special points lie outside the border basis scheme, not just outside the set $\Gamma(U)$. A case in point is Example 3.9 of [16] where the value $a = 0$ corresponds to an ideal I_0 which has no \mathcal{O} -border basis.

Our last example shows that the flat families we construct do, in general, not preserve the geometry of the corresponding sets of points.

Example 4.9. In the setting of the preceding example, we replace the scheme \mathbb{Y} by $\mathbb{Y}' = \{(0, 0), (-1, 1), (1, -1), (1, 2)\}$. Arguing as above, we see that \mathbb{Y}' is represented in \mathbb{A}^8 by the point $p_3 = (2, 2, 1, 1, -1, -1, 0, 0)$. Now we connect p_1 and p_3 by a line such that p_1 corresponds to the parameter value $a = 0$ and p_3 corresponds to $a = 1$. The resulting family of ideals uses

$$c_{21} = c_{23} = 2, c_{32} = c_{34} = a, c_{41} = c_{42} = -a, c_{43} = c_{44} = 1 - a$$

and its border bases are parametrically given by

$$\begin{aligned} H_a = & \{y^2 - 2x + (a^2 - 2a - 1)y + axy, \\ & x^2 - x - ay + axy, \\ & xy^2 - 2x + (a^2 - 2a)y - (1 - a)xy, \\ & x^2y - ay - (1 - a)xy\} \end{aligned}$$

The ideal J_a generated by H_a is reduced for every a . For $a = 0$, the ideal $J_0 = \mathcal{I}(\mathbb{X})$ corresponds to a set of four points in general position, but for $a = 1$ the ideal $J_1 = \mathcal{I}(\mathbb{Y}')$ corresponds to four points, three of which lie on the line $\mathcal{Z}(x + y)$. In

geometrical jargon one can express this by stating that the set \mathbb{X} has the Cayley-Bacharach property, but \mathbb{Y}' doesn't. In this sense the flat family did not preserve the geometry of the point set.

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